Ejercicio cálculo máximo común divisor a través de la factorización de polinomios.

Dados los polinomios,

$$p(x) = 2x^5+16x^4+29x^3-8x^2-15x \quad \text{y} \quad q(x) = x^2\,(2x^2-1)$$

Deducir, a través de la factorización en irreducibles de los polinomios $p(x)$ y $q(x)$ en $\mathbb{Z}[x]$, $\mathbb{Q}[x]$ y $\mathbb{Z}_{11}[x]$, si el máximo común divisor de ambos, en los anteriores anillos de polinomios es $(6x^3 - 3x)$.

$$p(x) = x\,(\underbrace{2x^4+16x^3+29x^2-8x-15}_{q(x)})$$

$$D_{15} = \{\pm 1, \pm 3, \pm 5, \pm 15\}$$

$$\begin{array}{r|rrrr|r}
 & 2 & 16 & 29 & -8 & -15 \\
-3 & \downarrow & -6 & -30 & 3 & 15 \\
\hline
 & 2 & 10 & -1 & -5 & \boxed{0} \\
-5 & \downarrow & -10 & 0 & 5 & \\
\hline
 & 2 & 0 & -1 & \boxed{0} &
\end{array}$$

$$D_5 = \{\pm 1, \pm 5\}$$

$$2x^2 - 1 = 0 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\sqrt{\frac{1}{2}}$$

$$\text{Raices de } p(x) = \{0, -3, -5, \underbrace{\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}}_{\notin \mathbb{Q}}\}$$

Fact. en $\mathbb{Q}[x]$

$$p(x) = 2x\,(x+3)\,(x+5)\left(x^2 - \frac{1}{2}\right)$$

Fact. en $\mathbb{Z}[x]$

$$p(x) = x\,(x+3)\,(x+5)\,(2x^2-1)$$

Fact. en $\mathbb{Z}_{11}[x]$

$$p(x) = x\,(x+3)\,(x+5)\,(2x^2+10)$$

$$2x^2+10 = 2\underbrace{(x^2+5)}_{r(x)}$$

$$\mathbb{Z}_{11} = \{0, 1, \dots, 10\}$$

$r(0) = 5 \neq 0$

$r(1) = 6 \neq 0$

$r(2) = 9 \neq 0$

$r(3) = 14 = 3 \neq 0$

$r(4) = 19 = 8 \neq 0$

$\dots$

$r(10) = 105 = 6 \neq 0$

Fact. en $\mathbb{Z}_{11}[x]$

$$p(x) = 2x(x+3)(x+5)(x^2+5)$$

$$q(x) = x^2(2x^2-1)$$

$\mathbb{Q}[x] \rightsquigarrow q(x) = 2x^2\left(x^2-\frac{1}{2}\right)$

$\mathbb{Z}[x] \rightsquigarrow q(x) = x^2(2x^2-1)$

$\mathbb{Z}_{11}[x] \rightsquigarrow q(x) = 2x^2(x^2+5)$

En resumen:

	$\mathbb{Q}[x]$	$\mathbb{Z}[x]$	$\mathbb{Z}_{11}[x]$
$p(x)$	$2x(x+3)(x+5)\left(x^2-\frac{1}{2}\right)$	$x(x+3)(x+5)(2x^2-1)$	$2x(x+3)(x+5)(x^2+5)$
$q(x)$	$2x^2\left(x^2-\frac{1}{2}\right)$	$x^2(2x^2-1)$	$2x^2(x^2+5)$
m.c.d	$2x\left(x^2-\frac{1}{2}\right) = 2x^3-x$	$x(2x^2-1) = 2x^3-x$	$2x(x^2+5) = 2x^3+10x$

$3 \in \mathcal{U}(\mathbb{Q}) = \mathcal{U}(\mathbb{Q}[x]) \Rightarrow 3(2x^3-x) = 6x^3-3x$

$3 \notin \mathcal{U}(\mathbb{Z}) \Rightarrow 3(2x^2 - x) \neq 2x^2 - x$

En $\mathbb{Z}[x]$, $6x^2 - 3x$ no es el m.c.d $\{p(x), q(x)\}$

En $\mathbb{Z}_{11}[x]$

$$6x^3 - 3x = 6x^3 + 8x$$

$$3(2x^3 + 10x) = 6x^3 + 30x = 6x^3 + 8x$$

$$3 \in \mathcal{U}(\mathbb{Z}_{11}) = \mathcal{U}(\mathbb{Z}_{11}[x])$$

$6x^3 + 8x$ es m.c.d $\{p(x), q(x)\}$ en $\mathbb{Z}_{11}[x]$